Pergamon

*J. Appl. Maths Mechs*, Vol. 57, No. 4, pp. 691–701, 1993

0021–8928/93 $24.00+0.00

0021–8928(93)E0012-L

# FREQUENCY DISPERSION OF ELASTIC WAVES IN DISORDERED COMPOSITES†

G. A. SHATALOV

Moscow

(*Received* 2 *April* 1992)

A version of the non-local theory of disordered composites (of randomly disordered media in a broad sense), which is constructed in terms of effective characteristics, is proposed. The latter are functions of a wave vector and frequency which enables one to describe the effects of spatial and frequency dispersion in wave propagation. The propagation of elastic waves solely taking account of frequency dispersion is investigated since little attention has been paid to this question. Results are obtained in the strong dispersion approximation, which corresponds to the long-wave but high-frequency approximation.

SPATIAL and frequency (time) dispersion occurs when elastic waves propagate in composites. While there is a quite extensive literature on the first problem ([1–6] and others), the second problem has not been sufficiently investigated [7]. Media with weak spatial dispersion which obey one of the versions of the couple theories of elasticity [8, 9] have been considered in the papers mentioned, although in [5], which takes account of the non-local properties of a micro-inhomogeneous medium, no constraints whatsoever are imposed on the wavelength of the waves.

It is of interest to construct a successively non-local theory of disordered composites which contains the local (zeroth approximation) and couple theories (the first approximation) and enables one to treat the two forms of wave dispersion within the framework of a universal formalism. Only frequency dispersion is considered below. The problem of spatial dispersion can also be treated within the framework of the formalism proposed.

## 1. STATEMENT OF THE PROBLEM

We will consider a two-phase composite material based on a matrix in which inclusions of the second phase of approximately equiaxial form are randomly distributed. The coupling between the phases is assumed to be ideal and the phases themselves are assumed to be isotropic. The effective characteristics of the composite, which describe its dynamic elastic behaviour are to be determined.

Let us assume that, in the long wavelength approximation, the composite, as a macroscopically and statistically homogeneous material, complies with the non-local theory of elasticity. In the non-local theory, an arbitrary characteristic $a^*$ is an integral operator with respect to the spatial and time variables. The kernel of this operator is of the difference type, and its Fourier transform can be represented by a power series in the wave vector $\mathbf{k}$ and frequency $\omega$. For an

†*Prikl. Mat. Mekh.* Vol. 57, No. 4, pp. 104–113, 1993.

isotropic medium, in the expansion (henceforth summation is carried out over repeated indices)

$$a^{*}(\mathbf{k};\omega)=a_0+a_{2\,ij}k_ik_j+a_2'\omega^2+\ldots \tag{1.1}$$

the zeroth approximation $a_0$ determines the effective characteristic of the local theory while the subsequent terms determine the constants of couple theories and describe the effects of spatial and frequency dispersion. It is assumed in (1.1) that $\mathbf{k}$ and $\omega$ are independent variables. If it turns out during the calculations that the effective characteristics of the composite are constant, then the composite, as a macroscopically homogeneous medium, is described by the local theory of elasticity. If, however, the Fourier transforms of the effective characteristics have the form of expansion (1.1), the composite complies with the non-local theory.

Green's function of a composite medium in the case of a dynamic elastic problem will be determined below. It is known [10] that the Fourier transforms of Green's functions are identical for the local and non-local theories, only, in the first case, they are expressed in terms of the constants $a_0$ and, in the second case, in terms of the function $a^{*}(\mathbf{k};\ \omega)$. This enables one to formulate an algorithm for calculating the effective characteristics which is common to both types of theory.

It is as follows. Green's function of a dynamic elastic problem $\mathbf{G}(\mathbf{k};\ \omega)$ for a composite medium is found. The procedures of averaging and passing to the long wavelength limit are carried out. The resulting Green's function $\langle\mathbf{G}(\mathbf{k};\ \omega)\rangle$ describes a macroscopically and statistically homogeneous material. Its Green's function $\mathbf{G}^{*}(\mathbf{k};\ \omega)$ is known and expressed in terms of the effective characteristics $a^{*}(\mathbf{k};\ \omega)$. The equation

$$\langle\mathbf{G}(\mathbf{k};\omega)\rangle=\mathbf{G}^{*}(\mathbf{k};\omega) \tag{1.2}$$

then enables one to determine $A^{*}(\mathbf{k};\ \omega)$.

## 2. GREEN'S FUNCTION OF A COMPOSITE MEDIUM

The phase geometry of the medium is described by a random function $\Theta(\mathbf{x})$ equal to unity if the radius vector $\mathbf{x}$ falls within an inclusion and zero otherwise. For the arbitrary constant of the composite, we have

$$\begin{gathered}a(\mathbf{x})=a_1(1-\Theta(\mathbf{x}))+a_2\Theta(\mathbf{x})=\langle a\rangle+\Delta a\,\Delta\Theta(\mathbf{x})\\ \langle a\rangle=a_1(1-c)+a_2c;\quad \Delta a=a_2-a_1\\ c=\langle\Theta(\mathbf{x})\rangle=\frac{1}{V}\int_V\Theta(\mathbf{x})\,dV;\quad \Delta\Theta(\mathbf{x})=\Theta(\mathbf{x})-c\end{gathered} \tag{2.1}$$

The subscripts 1 and 2 indicate the matrix and the inclusions, respectively, $c$ is the volume fraction of the inclusions, $V$ is the volume of the sample, and the symbol $\Delta$ is only used as a difference operator.

The equation of motion of a composite medium in the displacements $\mathbf{u}$ has the form

$$\begin{gathered}(\Gamma_{0ij}+W_{ij})u_j=0\\ \Gamma_{0ij}=-\langle\rho\rangle\delta_{ij}\frac{\partial^2}{\partial t^2}+\langle c_{iljm}\rangle\frac{\partial^2}{\partial x_l\partial x_m}\\ W_{ij}=-\Delta\rho\,\Delta\Theta(\mathbf{x})\delta_{ij}\frac{\partial^2}{\partial t^2}+\Delta c_{iljm}\frac{\partial}{\partial x_l}\Delta\Theta(\mathbf{x})\frac{\partial}{\partial x_m}\end{gathered} \tag{2.2}$$

Here $\delta_{ij}$ is the Kronecker delta, $\rho$ is the density, and $c_{iljm}$ is the elasticity tensor which, in the case of an isotropic medium, is equal to

$$c_{iljm} = \lambda\,\delta_{il}\,\delta_{jm} + \mu\,(\delta_{ij}\,\delta_{lm} + \delta_{im}\,\delta_{jl}) \tag{2.3}$$

where $\lambda$ and $\mu$ are Lamé constants.

Green's function $G_{ij}(\mathbf{x},\mathbf{x}';t)$ of Eq. (2.2) is determined from the relationship

$$(\Gamma_{0\,ij} + W_{ij})\,G_{jl}(\mathbf{x},\mathbf{x}';t) = -\delta_{il}\,\delta\,(\mathbf{x}-\mathbf{x}')\,\delta\,(t) \tag{2.4}$$

The boundary conditions to Eq. (2.4) will be discussed below.

We change from the coordinate–time representation $(\mathbf{x};\,t)$ to the frequency–wave representation $(\mathbf{k};\,\omega)$ using the formulae

$$\begin{aligned} f(\mathbf{k};\omega) &= \int\!\!\int_V f(\mathbf{x};t)\exp\,[-i\,(\mathbf{kx}-\omega\,t)]\,dVdt \\ f(\mathbf{x};t) &= \frac{1}{2\pi}\int\frac{1}{V}\sum_{k} f(\mathbf{k};\omega)\exp\,[i\,(\mathbf{kx}-\omega\,t)]\;d\,\omega \end{aligned} \tag{2.5}$$

since media based on a periodic structure will be considered. Equation (2.4) leads to the following integral equation for Green's function

$$\mathbf{G}(\mathbf{k},\mathbf{k}';\omega) = V\,\mathbf{G}_0(\mathbf{k};\omega)\,\delta_{\mathbf{kk}'} + \frac{1}{V}\sum_{\mathbf{k}_1}\mathbf{G}_0(\mathbf{k};\omega)\,\mathbf{W}(\mathbf{k},\mathbf{k}_1;\omega)\,\mathbf{G}(\mathbf{k}_1,\mathbf{k}';\omega) \tag{2.6}$$

Here $V\mathbf{G}_0(\mathbf{k};\,\omega)\delta_{\mathbf{kk}'}$ is Green's function for a dynamic problem in the theory of elasticity for a homogeneous medium characterized by physical constants $\langle a\rangle$ defined using the rule of mixtures (the second relationship of (2.1)). The expression for $\mathbf{G}_0(\mathbf{k};\,\omega)$ is obtained from the equation

$$\Gamma_{0\,ij}(\mathbf{k};\omega)\,G_{0\,jl}(\mathbf{k};\omega) = -\delta_{il} \tag{2.7}$$

Here

$$\begin{aligned} \Gamma_{0\,ij}(\mathbf{k};\omega) &= (\langle\mu\rangle k^2 - \langle\rho\rangle\omega^2)\,\delta_{ij} + \langle\lambda+\mu\rangle k_i k_j \\ G_{0\,ij}(\mathbf{k};\omega) &= (\langle\mu\rangle k^2 - \langle\rho\rangle\omega^2)^{-1}\,(\delta_{ij} - \Pi\,(\mathbf{k};\omega)\,n_i n_j) \\ \Pi\,(k;\omega) &= \frac{\langle\lambda+\mu\rangle k^2}{\langle\lambda+2\mu\rangle k^2 - \langle\rho\rangle\omega^2}, \quad n_i = \frac{k_i}{k} \end{aligned} \tag{2.8}$$

The tensor

$$\begin{aligned} W_{ij}(\mathbf{k},\mathbf{k}';\omega) &= (\Delta\rho\,\omega^2\,\delta_{ij} - \Delta\,c_{iljm}\,k_l\,k'_m)\,\Delta\,\Theta\,(\mathbf{k}-\mathbf{k}') = \\ &= W'_{ij}(\mathbf{k},\mathbf{k}';\omega)\,\Delta\,\Theta\,(\mathbf{k}-\mathbf{k}') \end{aligned} \tag{2.9}$$

may be referred to as the perturbation tensor since it contains the differences in the constants $\Delta a$ and information on the shape and relative arrangement of the inclusions which is included in $\Delta\Theta(\mathbf{k}-\mathbf{k}')$.

Equation (2.6) is referred to as an equation of the Dyson type, and its solution, obtained iteratively, has the form

$$\begin{aligned} \mathbf{G}(\mathbf{k},\mathbf{k}';\omega) &= \mathrm{V}\,G_0(\mathbf{k};\omega)\,\delta_{\mathbf{kk}'} + \mathbf{G}_0(\mathbf{k};\omega)\,\mathbf{W}(\mathbf{k},\mathbf{k}';\omega)\,\mathbf{G}_0(\mathbf{k}';\omega) + \\ &+ \frac{1}{V}\sum_{\mathbf{k}_1}\mathbf{G}_0(\mathbf{k};\omega)\,\mathbf{W}(\mathbf{k},\mathbf{k}_1;\omega)\,\mathbf{G}_0(\mathbf{k}_1,\omega)\,\mathbf{W}(\mathbf{k}_1,\mathbf{k}';\omega)\,\mathbf{G}_0(\mathbf{k}';\omega) + \ldots \end{aligned} \tag{2.10}$$

## 3. MODEL OF A COMPOSITE MATERIAL. AVERAGING OF GREEN'S FUNCTION

Up to now, the treatment has been carried out within the framework of the local theory of elasticity. The non-local character manifests itself in the theory at the stage when series (2.10) is averaged [11].

A sequential non-local theory must contain characteristic scales of the length parameter which are small compared with the characteristic dimensions of the body [10]. On the other hand, the treatment of wavelengths, commensurate with the scale parameter, which may be the characteristic size of an inclusion, is permitted in the theory. Distances smaller than the scale parameter are excluded from the treatment. Hence, on the one hand, the theory must be continuous and, on the other hand, it must have a "lattice" character. Such a theory can be constructed using the concept of a quasicontinuum [10].

The model of a composite material, where the idea of a quasicontinuum is realized, is constructed in the following manner. Let $R_0$ be the mean size of an inclusion and let there be $N$ cubes with a volume $v_0 = R_0^3$. Moreover, $n$ of these cubes are made of the material of the inclusion while the remaining $N-n$ cubes are made of the matrix material. Just like a crystal is constructed from unit cells, we construct the composite medium from hatched and unhatched cubes by arranging them in a random manner. As a result, one obtains one of the forms of a stochastic medium in a periodic lattice, which the two-dimensional spatial lattice of squares (the hatched squares are the inclusions and the unhatched squares are the matrix) shown in Fig. 1 illustrates. The nodal points of the lattice, which are indicated by the open circles, are found from the relationship

$$\mathbf{R} = m_i \mathbf{R}_{0i} \tag{3.1}$$

where $m_i$ are integers and $\mathbf{R}_{0i}$ is a vector along the Cartesian axis $i$ which has a length $R_0$.

The significance of the idea of a quasicontinuum involves the establishment of a one-to-one correspondence between functions with a discrete argument and a certain class of continuous functions. This correspondence is realizable [10] if the continuous functions can be represented as the superposition of long waves characterized by wave vectors $\mathbf{k}$ lying within the range

$$-\pi/R_0 \leqslant k_i \leqslant \pi/R_0 \tag{3.2}$$

In crystal physics, the domain (3.2) is referred to as the first Brillouin zone of the reciprocal lattice of the crystal. We shall henceforth retain this nomenclature assuming that it will not lead to any misunderstandings. It is obvious that the use of waves from the range (3.2) infers that short waves with lengths $\lambda < 2R_0$ are excluded from the treatment.

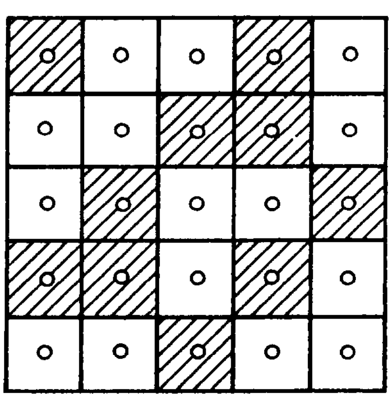

FIG. 1.

There is yet one more fact requiring elucidation. On a macroscale, the composite is statistically homogeneous. This means that the moments of various order of the random fields of the composite are independent of the absolute values of the coordinates but depend on their differences. For example, $\langle a(\mathbf{x}_1)a(\mathbf{x}_2)\rangle = f(\mathbf{x}_1 - \mathbf{x}_2)$. In the case of a sample with finite dimensions, this condition is violated in a narrow boundary layer. In order to avoid the effect of the narrow surface layer, we adopt so-called cyclic boundary conditions [12]. Under these conditions, the transition from the $(\mathbf{x}; t)$-representation to the $(\mathbf{k}; \omega)$-representation and the inverse transition are made using formulae (2.5). Here, the vector $\mathbf{k}$ lies in the domain (3.2).

The averaging of series (2.10) is carried out on the lattice (3.1). Let us introduce the random numbers $\eta(\mathbf{R}_\alpha)$, which are equal to unity if the centre of gravity of an inclusion is located at the nodal point of the lattice $\mathbf{R}_\alpha$ and zero otherwise. Then, describing the shape of an isolated inclusion found at $\mathbf{R}_\alpha$ by the deterministic function $\Theta_0(\mathbf{x}-\mathbf{R}_\alpha)$, we find

$$\Delta\Theta(\mathbf{x}) = \sum_{\alpha=1}^{N} \Delta\eta(\mathbf{R}_\alpha)\Theta_0(\mathbf{x}-\mathbf{R}_\alpha)$$

$$\Delta\Theta(\mathbf{k}-\mathbf{k}') = \sum_{\alpha=1}^{N} \Delta\eta(\mathbf{R}_\alpha)\Theta_0(\mathbf{k}-\mathbf{k}')\exp[-i(\mathbf{k}-\mathbf{k}')\mathbf{R}_\alpha] \tag{3.3}$$

$$\Delta\eta(\mathbf{R}_\alpha) = \eta(\mathbf{R}_\alpha) - \langle\eta(\mathbf{R}_\alpha)\rangle$$

Up to now the theory has been constructed taking account of both spatial and frequency dispersion. We note that the spatial dispersion may be attributable either to the microstructure of the medium (the physical non-local character) or to the finite size of the inhomogeneity (the geometric non-local character) [10]. At this stage, we exclude the spatial dispersion associated with the finite size of an inclusion by assuming the inclusions to be "point" inclusions and set

$$\Theta(\mathbf{x}-\mathbf{R}_\alpha) = v_0\,\delta(\mathbf{x}-\mathbf{R}_\alpha), \quad \Theta_0(\mathbf{k}-\mathbf{k}'=0) = v_0$$

We average (2.10) over the ensemble of systems, each being one of the forms of stochastic medium on the lattice (3.1) similar to that shown in Fig. 1. All the systems of the ensemble are equiprobable. Then

$$\langle\eta(\mathbf{R}_\alpha)\rangle = \frac{1}{N}\sum_{\alpha=1}^{N}\eta(\mathbf{R}_\alpha) = \frac{n}{N} = \frac{n\,v_0}{N\,v_0} = c$$

As can be seen from the structure of the series (2.10), products of the type $\Delta\eta(\mathbf{R}_\alpha)\Delta\eta(\mathbf{R}_\beta)\ldots\Delta\eta(\mathbf{R}_\lambda)$ are subjected to averaging. The zeroth term of the series does not contain these numbers. After averaging, the first term of the series disappears as $\langle\Delta\eta(\mathbf{R}_\alpha)\rangle = 0$. When determining $\langle\Delta\eta(\mathbf{R}_\alpha)\Delta\eta(\mathbf{R}_\beta)\rangle$ in the second term, it is necessary to take into account whether $\mathbf{R}_\alpha$ and $\mathbf{R}_\beta$ are identical or not. If they are not then, by virtue of statistical independence

$$\langle\Delta\eta(\mathbf{R}_\alpha)\Delta\eta(\mathbf{R}_\beta)\rangle(1-\delta_{\alpha\beta}) = \langle\Delta\eta(\mathbf{R}_\alpha)\rangle\langle\Delta\eta(\mathbf{R}_\beta)\rangle(1-\delta_{\alpha\beta}) = 0$$

If they are, then

$$\langle\Delta\eta(\mathbf{R}_\alpha)\Delta\eta(\mathbf{R}_\beta)\rangle\delta_{\alpha\beta} = \langle(\Delta\eta(\mathbf{R}_\alpha))^2\rangle = \frac{1}{N}\sum_{\alpha=1}^{N}(\eta(\mathbf{R}_\alpha)-c)^2 = c(1-c)$$

When averaging the $n$th term of the series, one has to take account of all cases of the partitioning of a set of $n$ points into all possible subsets. Within the limits of each subset, consisting of $m$ points, all the arguments $\mathbf{R}_{\alpha_i}$ $(i=1, 2, \ldots, m)$ are identical. The corresponding calculations, which enable one to carry out the accurate retention of all the terms of series (2.10) taking account of the fact that, during averaging, each of them is decomposed into a certain number of terms, can be accomplished using a diagrammatic technique [13].

Analysis shows that a certain class of diagrams which are characterized by the intersection of the correlation lines, is described by analytic expressions containing polynomials in powers of the wave vector $k$ even in the limit of "point" inclusions. These diagrams then describe the contribution of the spatial dispersion due to the microstructure of the medium. Excluding these diagrams from the treatment, we make use of the results of the summation of the infinite subsequence of all diagrams without intersection of the correlation lines [13]. Green's function corresponding to this subsequence has the form $V\langle \mathbf{G}(\mathbf{k};\ \omega)\rangle \delta_{\mathbf{kk}'}$, where $\langle \mathbf{G}(\mathbf{k};\ \omega)\rangle$ is the solution of the algebraic equation

$$
\begin{aligned}
&\langle \mathbf{G}(\mathbf{k};\omega)\rangle = \mathbf{G}_0(\mathbf{k};\omega) + \mathbf{G}_0(\mathbf{k};\omega)\,\mathbf{R}(\mathbf{k};\omega)\langle \mathbf{G}(\mathbf{k};\omega)\rangle \\
&\mathbf{R}(\mathbf{k};\omega) = \sum_{n=2}^{\infty} \chi_n \mathbf{W}_n(\mathbf{k},\mathbf{k};\omega) \\
&\mathbf{W}_n(\mathbf{k},\ \mathbf{k}_n;\omega) = \left(\frac{v_0}{V}\right)^{n-1} \sum_{\mathbf{k}_1\ldots\mathbf{k}_{n-1}} \mathbf{W}'(\mathbf{k},\mathbf{k}_1;\omega)\,\mathbf{G}_0(\mathbf{k}_1;\omega)\,\mathbf{W}'(\mathbf{k}_1,\mathbf{k}_2;\omega)\ldots \\
&\ldots \mathbf{G}_0(\mathbf{k}_{n-1};\omega)\,\mathbf{W}'(\mathbf{k}_{n-1},\mathbf{k}_n;\omega)
\end{aligned}
\tag{3.4}
$$

and the tensor $\mathbf{W}'(\mathbf{k}, \mathbf{k}, \mathbf{k}';\ \omega)$ is defined by formula (2.9). The quantities $\chi_n$ are the $n$th order cumulants of a random quantity which takes two values: $1-c$ with probability $c$ and $-c$ with probability $1-c$. The cumulants $\chi_n$ are as follows [14]:

$$
\chi_n = \sum_{s=1}^{[n/2]} \frac{(-1)^{s-1}}{s}\, C_{2s-2}^{s-1}\, C_{n-2}^{2s-2}\, c^s (1-c)^s (1-2c)^{n-2s}
\tag{3.5}
$$

and have the generating function

$$
\chi(t) = \sum_{n=2}^{\infty} \chi_n t^n = -\tfrac{1}{2}\left(1-\psi t-(1-2\psi t+t^2)^{1/2}\right), \quad \psi = 1-2c
\tag{3.6}
$$

## 4. DETERMINATION OF THE EFFECTIVE CHARACTERISTICS

Using (1.2) and the fact that, in the case of a homogeneous medium, the functions $\mathbf{G}_0(\mathbf{k};\ \omega)$ and $\Gamma_0(\mathbf{k};\ \omega)$ are related by Eq. (2.7), let us rewrite Eq. (3.4) in the form

$$
\Gamma^*(\mathbf{k};\omega) = \Gamma_0(\mathbf{k};\omega) + \mathbf{R}(\mathbf{k};\omega)
\tag{4.1}
$$

The function $\Gamma_0(\mathbf{k};\ \omega)$ is defined by the first equation in (2.8) and $\Gamma^*(\mathbf{k};\ \omega)$ is defined by the same equation in which, however, $a^*(\mathbf{k};\ \omega)$ are substituted for the characteristics $\langle a\rangle$, determined using the rule of mixtures. The problem therefore reduces to calculating the tensor $\mathbf{R}(\mathbf{k};\ \omega)$.

The tensors $\mathbf{W}_n$ satisfy a recurrence equation with the initial condition

$$
\mathbf{W}_{n+1}(\mathbf{k},\mathbf{k}_{n+1};\omega) = \frac{v_0}{V}\sum_{\mathbf{k}_n} \mathbf{W}_n(\mathbf{k},\mathbf{k}_n;\omega)\,\mathbf{G}_0(\mathbf{k}_n;\omega)\,\mathbf{W}'(\mathbf{k}_n,\mathbf{k}_{n+1};\omega)
\tag{4.2}
$$

$$
\mathbf{W}_1(\mathbf{k},\mathbf{k}_1;\omega) = \mathbf{W}'(\mathbf{k},\mathbf{k}_1;\omega)
\tag{4.3}
$$

Putting

$$C_{ijlm} = -\Delta\, c_{ijlm} = 3\, K V_{ijlm} + 2\, M D_{ijlm}$$
$$K = -\Delta\,\lambda - {}^2\!/_3\, \Delta\,\mu, \quad M = -\Delta\,\mu \tag{4.4}$$

where $V_{ijlm}$ and $D_{ijlm}$ are the volume and deviator components of the fourth-rank unit tensor respectively, we write (2.9) in the form

$$W'_{ij}(\mathbf{k},\mathbf{k}';\omega) = P\,\omega^2 \delta_{ij} + C_{iljm}\, k_l\, k'_m, \quad P = -\Delta\,\rho \tag{4.5}$$

We will assume that

$$W^{(n)}_{ij}(\mathbf{k},\mathbf{k}_n;\omega) = P_n\,\omega^2 \delta_{ij} + C^{(n)}_{iljm}\, k_l\, k^{(n)}_m \tag{4.6}$$

$$C^{(n)}_{iljm} = 3\, K_n\, V_{iljm} + 2\, M_n\, D_{iljm} \tag{4.7}$$

and prove this representation by induction. It is obvious that the formulae hold when $n=1$ and that $P_1 = P$, $K_1 = K$, $M_1 = 1$. We now substitute expressions (4.5) and (4.6) and the second relationship of (2.8) into (4.2) and replace the sum in (4.2) by an integral according to the rule [12]

$$\frac{1}{V}\sum_{\mathbf{k}} \ldots \rightarrow \int \ldots \frac{d^3\mathbf{k}}{(2\,\pi)^3}$$

where the integration is carried out over the first Brillouin zone (3.2) which has a volume $(2\pi)^3/v_0$. This integration is replaced by integration over a sphere of equal volume $4\pi\tau_0^3/3$ (the Debye model of a quasicontinuum which is applicable to isotropic media [10]). The integrals which arise here exist in the sense of the principal value. Integrating, we arrive at formulae (4.6) and (4.7) with $n$ replaced by $n+1$

$$P_{n+1} = \alpha P_n, \quad \alpha = (P/\langle\rho\rangle)\,(2\,B\,(\langle z_t\rangle) + B\,(\langle z_l\rangle))$$
$$K_{n+1} = \beta K_n, \quad \beta = (K/\langle\lambda + 2\,\mu\rangle)\,(1 + 3\,B\,(\langle z_l\rangle))$$
$$M_{n+1} = \gamma M_n, \quad \gamma = \frac{2\,M}{5\,\langle\mu\rangle}\left(\frac{\langle 3\,\lambda + 8\,\mu\rangle}{\langle\lambda + 2\,\mu\rangle} + 3\,B\,(\langle z_t\rangle) + \frac{2\,\langle\mu\rangle}{\langle\lambda + 2\,\mu\rangle} B\,(\langle z_l\rangle)\right) \tag{4.8}$$
$$B\,(z) = z^2\left(1 + \frac{1}{2}\, z\, \ln\frac{1-z}{1+z}\right)$$
$$\langle z_t\rangle = \frac{\omega}{\tau_0}\left(\frac{\langle\rho\rangle}{\langle\mu\rangle}\right)^{1/2}, \quad \langle z_l\rangle = \frac{\omega}{\tau_0}\left(\frac{\langle\rho\rangle}{\langle\lambda + 2\,\mu\rangle}\right)^{1/2}, \quad \tau_0 = \left(\frac{3}{4\,\pi}\right)^{1/3}\frac{2\,\pi}{R_0}$$

Equalities (4.8) yield recurrence relationships for the quantities occurring in (4.6) and (4.7), where

$$P_n = P\,\alpha^{n-1}, \quad K_n = K\,\beta^{n-1}, \quad M_n = M\,\gamma^{n-1} \tag{4.9}$$

Calculating the function $\mathbf{W}_n(\mathbf{k}, \mathbf{k}_n; \omega)$ from (3.4) using (4.6), (3.5) and (3.6), we find

$$R_{ij}\,(\mathbf{k};\omega) = (Mk^2\,\chi\,(\gamma)/\gamma + P\,\omega^2\,\chi\,(\alpha)/\alpha)\,\delta_{ij} +$$
$$+ \;(K\,\chi\,(\beta)/\beta + {}^1\!/_3\, M\,\chi\,(\gamma)/\gamma)\, k_i\, k_j \tag{4.10}$$

Finally, using (4.1), (4.10) and (2.8), we obtain the system for determining the effective characteristics

$$\begin{aligned}\mu^* &= \langle\mu\rangle - M\chi(\gamma)/\gamma \\ \lambda^* + \mu^* &= \langle\lambda+\mu\rangle - K\chi(\beta)/\beta - {}^1\!/_3\, M\chi(\gamma)/\gamma \\ \rho^* &= \langle\rho\rangle + P\chi(\alpha)/\alpha\end{aligned} \tag{4.11}$$

The effective characteristics are determined [13] from systems of the type of (4.11) using the iterative procedure which we will use below in the numerical solution. The mean characteristics $\langle a\rangle$ are initially substituted into expressions of the type $\chi(\alpha)/\alpha$ and the characteristics in the first approximation are then calculated. Next, they are again substituted into $\chi(\alpha)/\alpha$, instead of $\langle a\rangle$, and the characteristics in the second approximation are obtained. The process is then repeated. On some, generally speaking, infinite step, the output characteristics cease to differ from the input characteristics. They are then accepted as the true values of the effective characteristics. Under these conditions, system (4.11) has the form

$$\begin{aligned}1+3B(z_l^*) &= (K^*+\frac{4}{3}\mu^*)\left(\frac{c}{K^*-K_1}+\frac{1-c}{K^*-K_2}\right) \\ 1+3B(z_t^*)+\frac{2\mu^*}{3K^*+4\mu^*}(1+3B(z_l^*)) &= \frac{5}{2}\mu^*\left(\frac{c}{\mu^*-\mu_1}+\frac{1-c}{\mu^*-\mu_2}\right) \\ 2B(z_t^*)+B(z_l^*) &= -\rho^*\left(\frac{c}{\rho^*-\rho_1}+\frac{1-c}{\rho^*-\rho_2}\right)\end{aligned} \tag{4.12}$$

($K$ is the bulk modulus of elasticity).

## 5. DISCUSSION OF THE RESULTS

The functions $B(z^*)$ which occur in system (4.12) depend solely on the frequency. The latter means that the effective characteristics $a^*(\mathbf{k};\,\omega)$ obtained as the solution of Eqs (4.12), are functions of the frequency and describe the frequency dispersion accompanying the propagation of elastic waves in a composite.

Frequency dispersion occurs in a medium during the occurrence of internal processes, the flow time of which is comparable to the period of the change in the external field. In this frequency domain, the response of the system to a change in the external field is retarded and the field in the medium at the given instant starts to depend on the applied field at the preceding instants of time. It is seen from the expression for $\langle z\rangle$ in (4.8) that the quantities $z_t^*$ and $z_l^*$ have the form $z^*=\omega t_0$, where $t_0 \simeq R_0/(\mu^*/\rho^*)^{1/2}$ or $t_0 \simeq R_0/((\lambda^*+2\mu^*)/\rho^*)^{1/2}$ are the times during which elastic waves travel the distance $R_0$.

The question arises as to the mechanism of the appearance of non-local inertial characteristics. The concept of "connected masses" [15], which is well known in hydrodynamics, may serve [10] as one of the sources of non-local behaviour.

Setting $\omega=0$ in (4.12), we find the system for the static effective characteristics

$$\begin{aligned}(K_0-K_1)(K_0-K_2) &= (K_0+{}^4\!/_3\,\mu_0)(K_0-\langle K\rangle) \\ (K_0+2\mu_0)(\mu_0-\mu_1)(\mu_0-\mu_2) &= {}^5\!/_2\,\mu_0(K_0+{}^4\!/_3\,\mu_,\ (\mu_0-\langle\mu\rangle) \\ \rho_0 &= \langle\rho\rangle\end{aligned} \tag{5.1}$$

Equations (5.1) are identical to the equations in the method of self-consistency [16].

In the limit of weak dispersion when

$$a^*(\mathbf{k};\omega) = a_0 + a_2'\omega^2$$

we find from (4.12) and (5.1) that

$$
\begin{aligned}
&3\,\rho_0/\tau_0^2 = K_2'(1-cA_1^2-(1-c)\,A_2^2) + {}^4/_3\ \ \mu_2' \\
&3\,\rho_0/\tau_0^2 = {}^2/_3\,(cB_1^2 + (1-c)\,B_2^2)\,K_2' + (1-{}^5/_2\,cC_1^2-{}^5/_2\,(1-c)\,C_2^2)\,\mu_2' \\
&\rho_2' = \frac{c\,(1-c)\,(\rho_2-\rho_1)^2}{\tau_0^2}\ \frac{2\,K_0 + {}^{11}/_3\,\mu_0}{\mu_0\,(K_0 + {}^4/_3\,\mu_,)} \\
&A_q = \frac{K_0 + {}^4/_3\,\mu_0}{K_0-K_q}, \quad B_q = \frac{\mu_0}{K_0-K_q}, \quad C_q = \frac{\mu_0}{\mu_0-\mu_q}, \quad q = 1, 2
\end{aligned}
\tag{5.2}
$$

It is better to use system (4.11) and the iterative procedure described above for the numerical solution. The constants for a glass–epoxy material were employed as the initial constants: $K_1 = 4.17\times10^9$ Pa, $\mu_1 = 0.9\times10^9$ Pa, $\rho_1 = 1.2\times10^3$ kg/m$^3$; $K_2 = 73.5\times10^9$ Pa, $\mu_2 = 29.4\times10^9$ Pa, $\rho_2 = 2.58\times10^3$ kg/m$^3$.

In the numerical solution, the iterative process was terminated when the relative difference between the input and the output became less than 0.1%. If the required accuracy had not been attained after $10^3$ iterations it was assumed that there was no solution. The results of the calculation are presented in Figs 2–5 for $c = 0.5$. Qualitatively similar results also hold for other values of $c$.

Figure 2 shows the dependence of the velocity of propagation of transverse elastic waves $\upsilon_t = (\mu^*/\rho^*)^{1/2}$ on the parameter $w = \omega/\tau_0 = (4\pi/3)^{1/3}\nu R_0$, where $\nu$ is the frequency of the wave. Using the condition $\lambda > 2R_0$, it can be shown that $z_t < \frac{1}{2}(4\pi/3)^{1/3} \simeq 0.806$, $z_l < 0.806$, that is, with the same scales along the coordinate axes, the plot $\upsilon_t = \upsilon_t(w)$ must lie above the bisector of the coordinate angle as is also shown in Fig. 2. The same also holds in the case of longitudinal waves.

Figures 3–5 show graphs of the dynamic effective moduli of elasticity $K^*$ and $\mu^*$ and the effective density $\rho^*$ as a function of the parameter $w$. The dashed lines correspond to the case of weak dispersion. We note the occurrence of a boundary value $w_0 = 1148$ m/s, above which no solution is found. In the case of coarsely dispersed media ($\tau_0$ is large), significant dispersion of the waves is displayed in a lower frequency domain than in the case of finely dispersed media. Finally, the numerical analysis showed that the plots shown in Figs 2–5 can be described by polynomials from $\omega^2$ up to $\omega^{10}$ or $\omega^{12}$. The latter means that the results are obtained in the strong dispersion approximation.

The results of the calculation shown in Figs 2–5 can be subjected to experimental verification. Here, it must be remembered that spatial and frequency dispersion as well as the dispersion due to visco-elastic effects simultaneously make a contribution to the experimental results. It is therefore initially necessary to evaluate the relative fraction of each of these contributions in each actual case. As far as comparison with known theoretical results is concerned, there is qualitative agreement in the limit of weak dispersion to which the majority of papers refer. For example, the dependence of the phase velocity of longitudinal waves on the frequency for the scattering of Rayleigh waves by a system of chaotically arranged inclusions is of the same character [17] as in the case under consideration. The analogous dependences for a regular laminated composite, for which an exact solution may be obtained, behave in the same manner at low frequencies.

In the limit of weak dispersion, it is possible to find the equation of the non-local theory of elasticity, which the composite obeys on a macroscale. This equation in the displacements has the form

$$
\begin{aligned}
&\rho_2'\,\partial^4\mathbf{u}/\partial t^4 + I\,\partial^2\mathbf{u}/\partial t^2 - \mu_0\nabla^2\mathbf{u} - (\lambda_0 + \mu_0)\,\mathrm{grad\,div}\,\mathbf{u} = \mathbf{q} \\
&I = \rho_0 - (\mu_2'\nabla^2 + (\lambda_2' + \mu_2')\,\mathrm{grad\,div})
\end{aligned}
\tag{5.3}
$$

($\mathbf{q}$ is the density of the external forces and $I$ is the operator of the inertial properties of the medium).

The proposed model of a composite medium and the algorithm for calculating the effective characteristics can be used for fields of an arbitrary physical nature.

The author thanks Yu. I. Karkovskii for discussions and help.

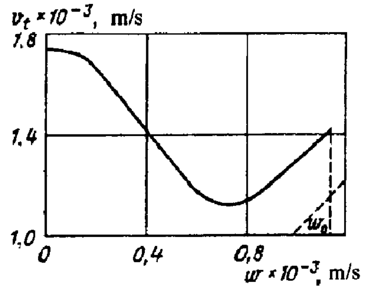


FIG. 2.

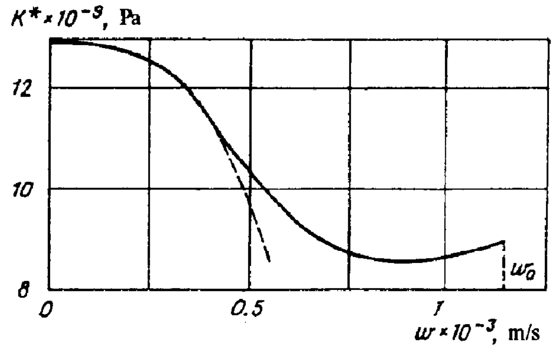


FIG. 3.

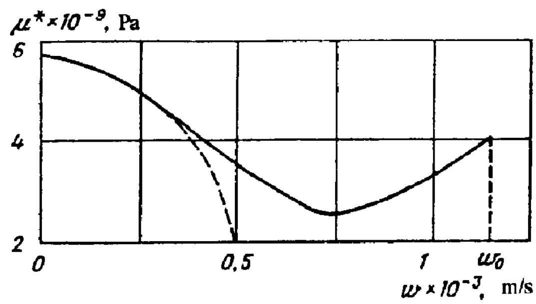


FIG. 4.

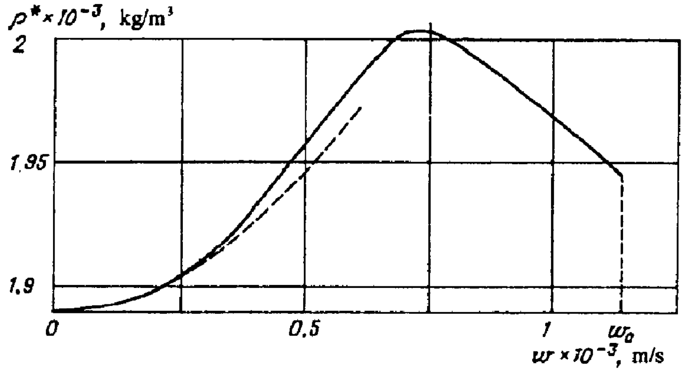


FIG. 5.

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*Translated by* E.L.S.